Image Processing

Discrete Fourier Transform

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Overview

Discrete Fourier Transform (DFT)

The spectrum of DFT

The convolution theorem

Filtering in the frequency domain

Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is described by the matrix

$$T = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

where $\omega = \mathrm{e}^{2i\pi/n} = \cos(2\pi/n) + i\sin(2\pi/n)$

2D DFT

The 2D DFT is given by $\boldsymbol{Z} = \boldsymbol{T} \cdot \boldsymbol{A} \cdot \boldsymbol{T}'$, where $\boldsymbol{T}$ is the matrix of the 1D DFT.

Using matrix notation we write in a single equation the computations for the whole transformation.

Instead, we could write an equation giving a single element of the transform:

$$Z(u,v) = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} A(x,y) e^{-2\pi i (ux+vy)/n}$$

2D DFT

The matrix $\boldsymbol{T}$ is complex, that is, its elements are complex numbers, not real numbers.

The same is generally true for the transform $\boldsymbol{Z}$ of an image. Its elements have a real part $\boldsymbol{R(u,v)}$ and an imaginary part $\boldsymbol{I(u,v)}$:

$$Z(u,v) = R(u,v) + i \cdot I(u,v)$$

So, generally, $\boldsymbol{Z}$ is not an image.

2D DFT

We can go around this problem by using real functions of the transform, that is, the spectrum:

$$\left|Z(u,v)\right| = \sqrt{R^2(u,v) + I^2(u,v)}$$

and the phase angle:

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

Usually, we visualize the DFT by visualizing its spectrum.

Overview

Discrete Fourier Transform (DFT)

The spectrum of the DFT

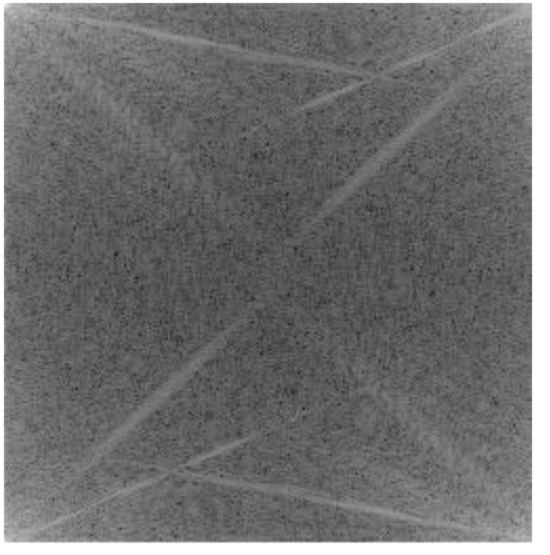
The convolution theorem

Filtering in the frequency domain

The spectrum of DFT

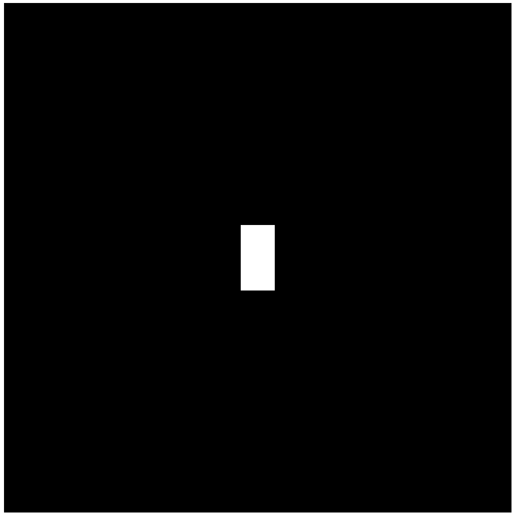
Original image

Log enhanced
Fourier transform



The spectrum of DFT

Original image



Fourier transform

The large coefficients concentrate on the four corners of the transform.

Low and high frequencies

Why are the low frequencies at the four corners of the transform?

As an example, we consider the DFT matrix of order 7.

We will visualize the elements of the matrix, which are complex numbers, as points on the plane.

We will visualize the rows of the matrix, which give the basis of the transform, by connecting these points with edges.

Low and high frequencies

The DFT matrix for *n*=7:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$

where $\omega = \mathrm{e}^{2i\pi/7} = \cos(2\pi/7) + i\sin(2\pi/7)$

Low and high frequencies

A complex number can be described by the pair (r,φ) where r is its distance from the origin and φ its angle with the *x*-axis.

If $z_1 = (r_1 , \varphi_1)$ and $z_2 = (r_2 , \varphi_2)$ are two complex numbers then:

$$z_1 \cdot z_2 = (r_1 \cdot r_2 , \varphi_1 + \varphi_2)$$

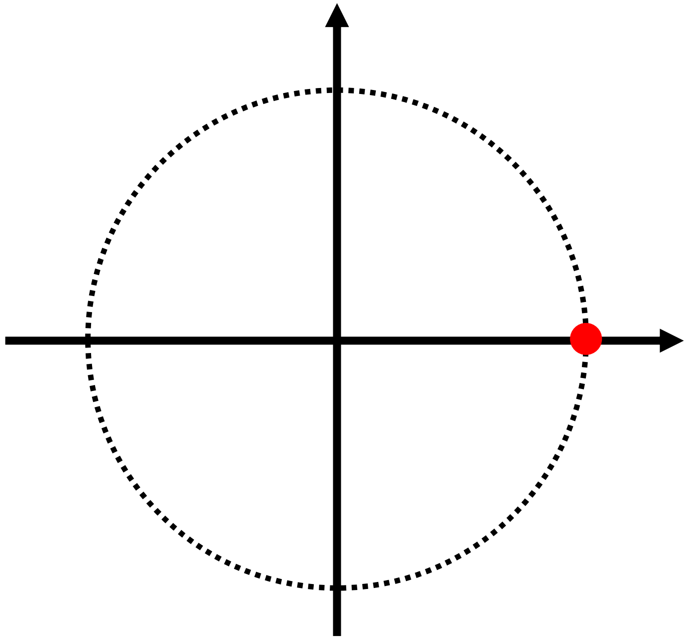
Here we have $\omega = (1 , 2\pi / 7)$, giving

$$\omega^n = (1 , 2\pi n / 7)$$

Low and high frequencies

The visualisation of the first row of the matrix (first element of the new basis) is trivial as there is only one point.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the second row.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\
1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\
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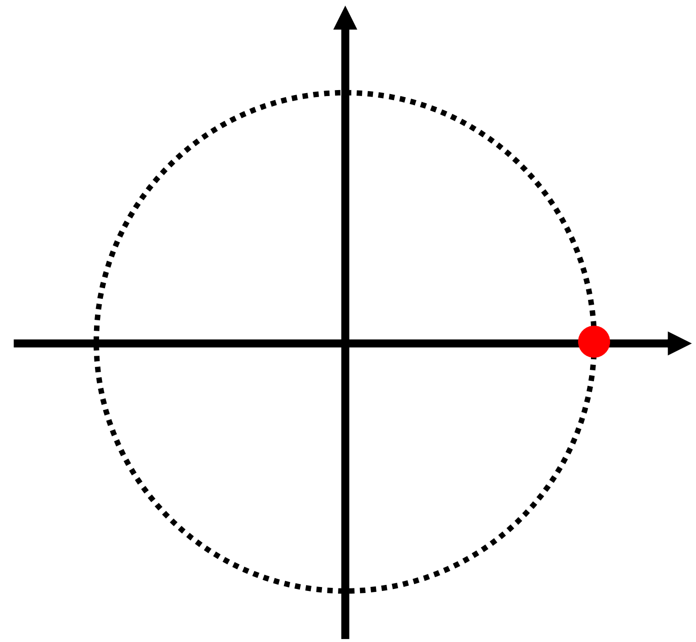
\end{bmatrix}
$$



Low and high frequencies

Visualisation of the second row.

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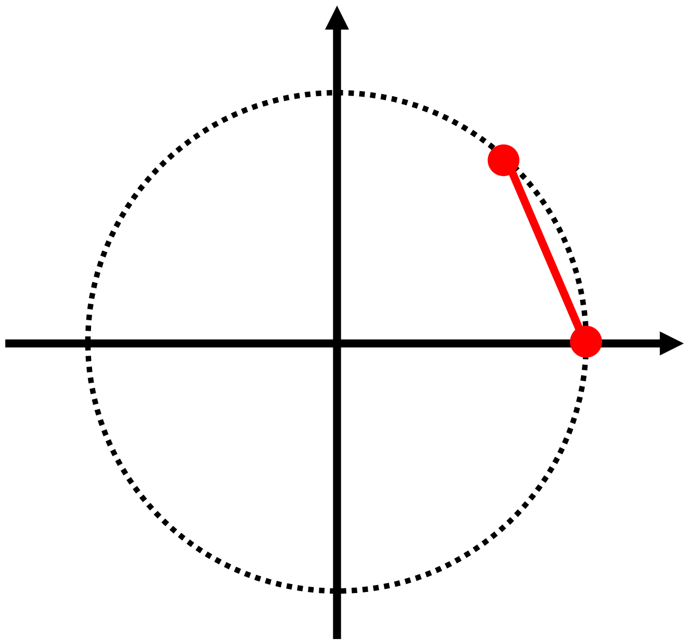

Low and high frequencies

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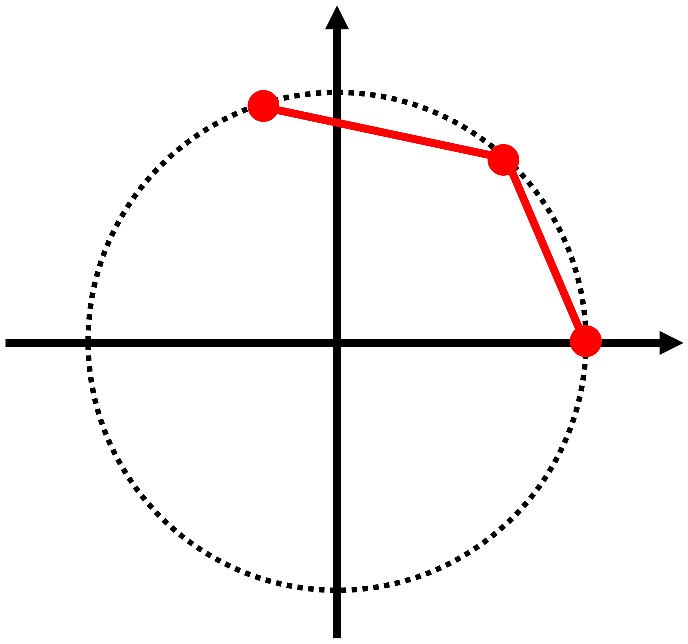

Low and high frequencies

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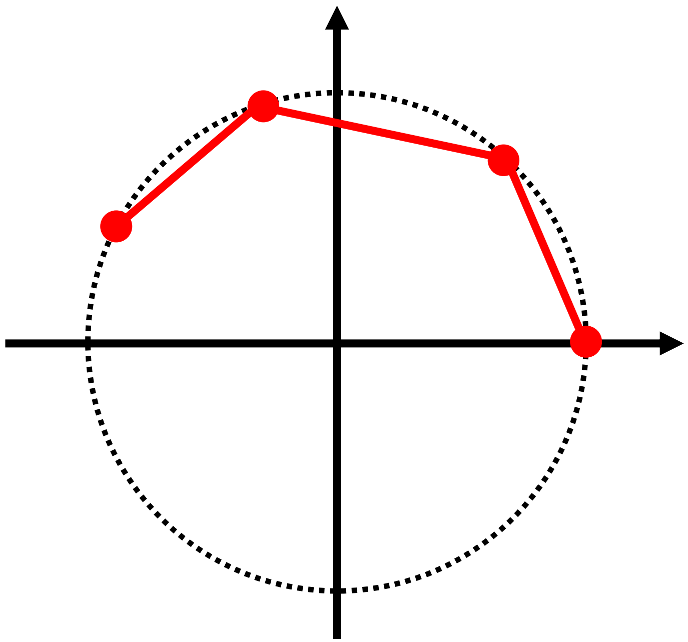
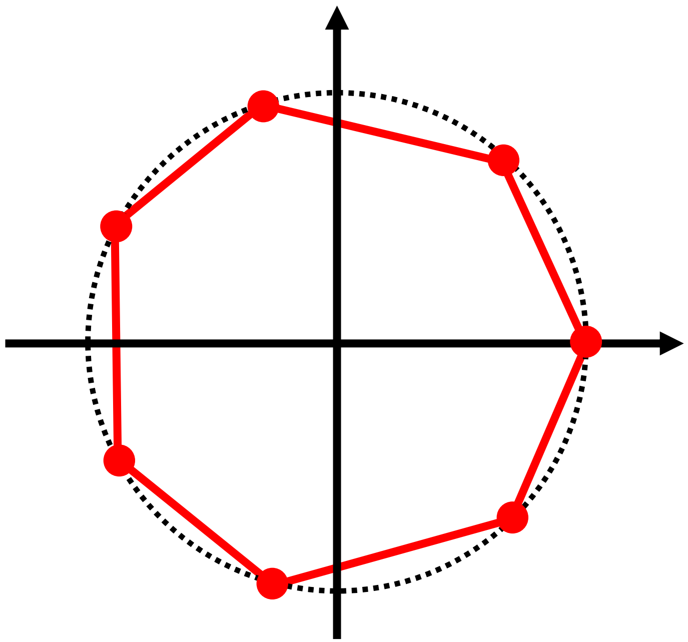


Low and high frequencies

Visualisation of the second row. We made one full anticlockwise circle.

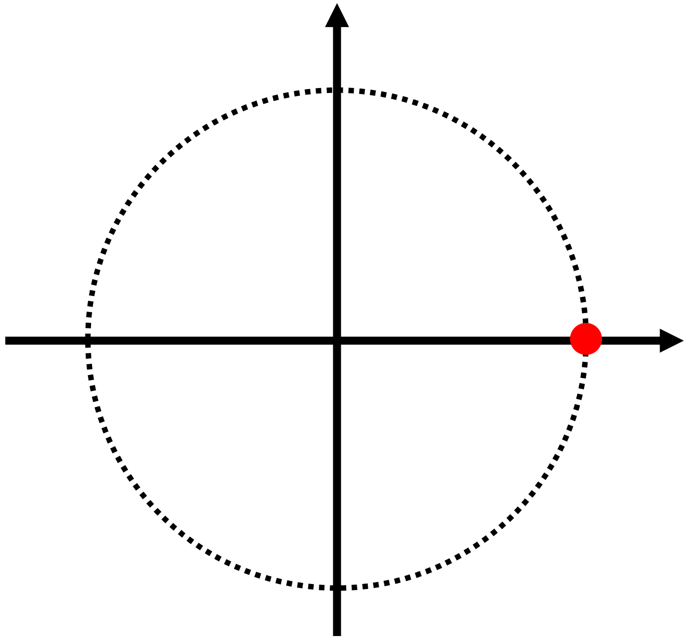
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

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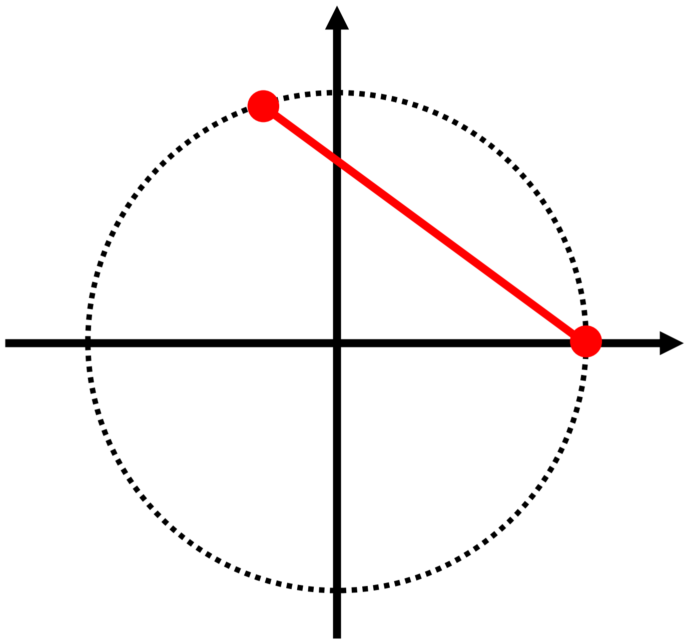

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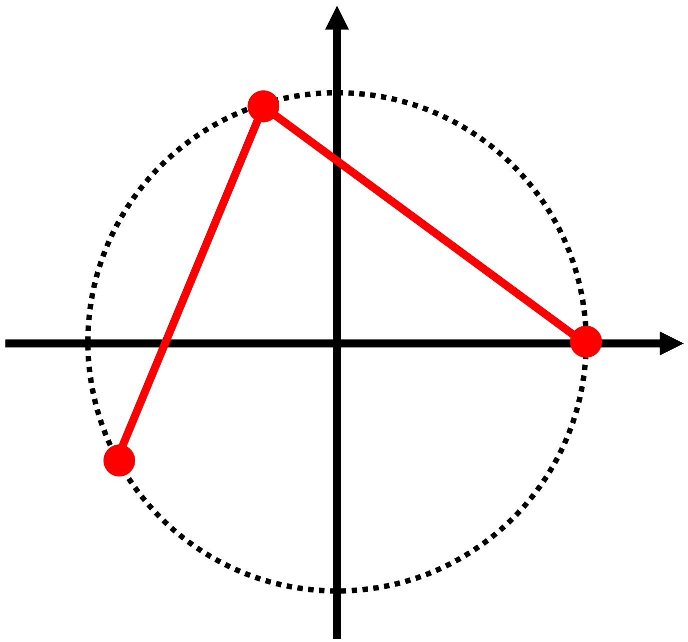

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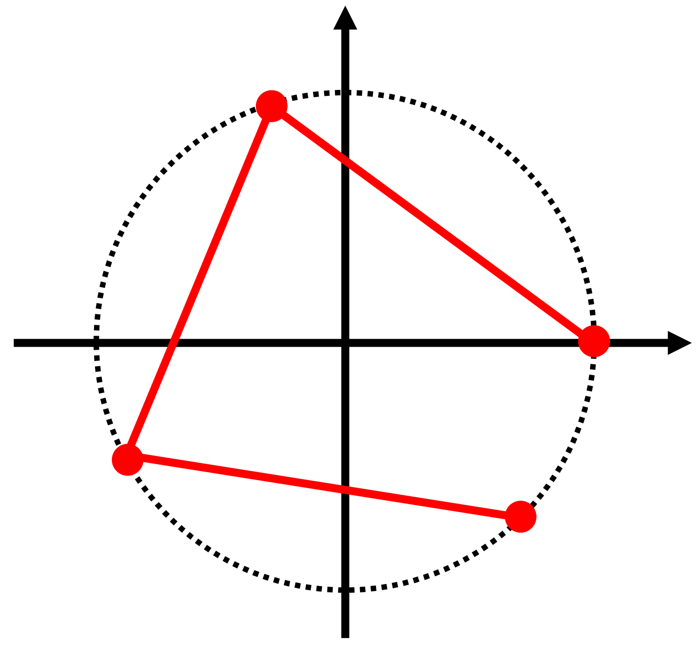

Low and high frequencies

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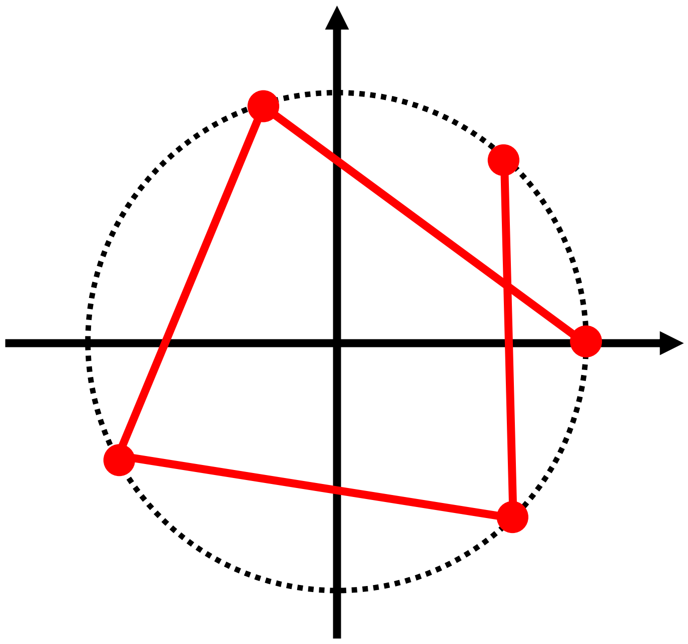
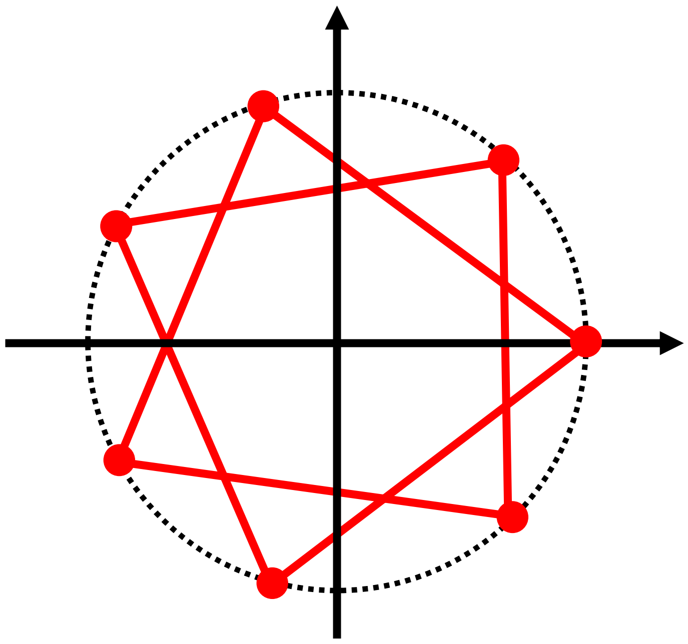


Low and high frequencies

Visualisation of the third row. We made two full anticlockwise circles.

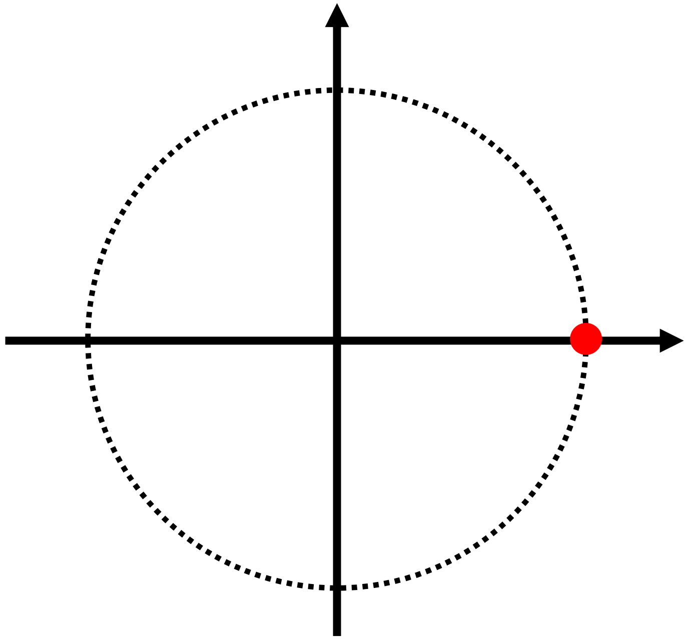
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the fourth row.

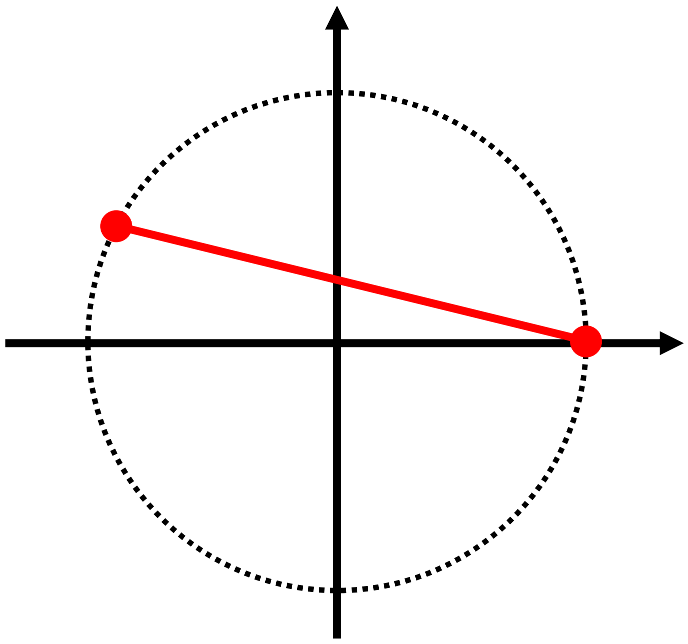
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the fourth row.

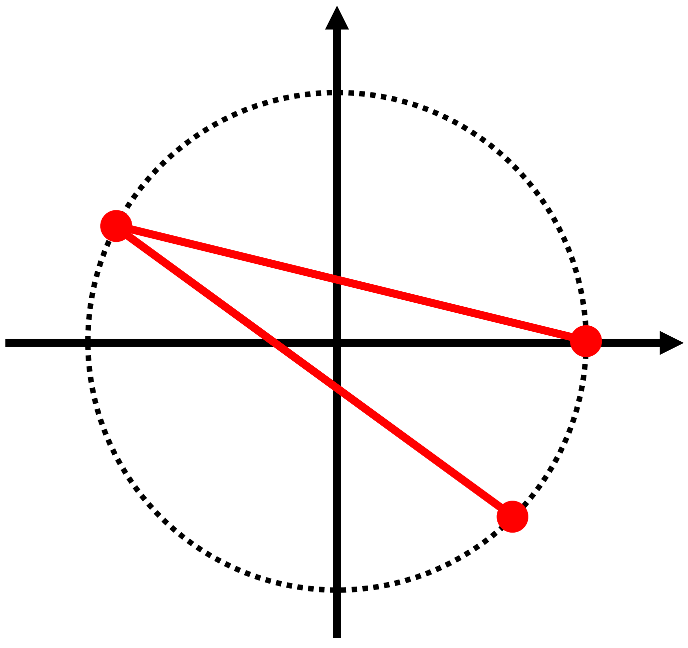
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the fourth row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & \omega^{8} & \omega^{10} & \omega^{12} \\ 1 & \omega^{3} & \omega^{6} & \omega^{9} & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^{4} & \omega^{8} & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^{5} & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^{6} & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

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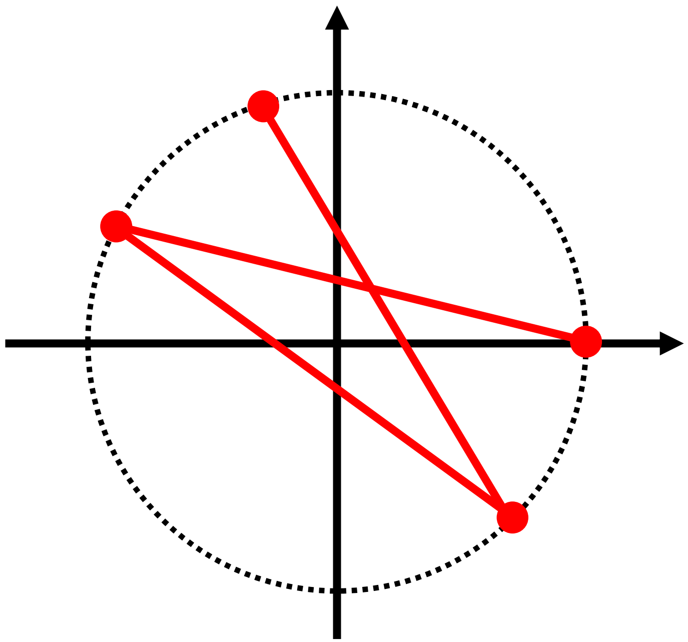

Low and high frequencies

Visualisation of the fourth row. We made three full anticlockwise circles.

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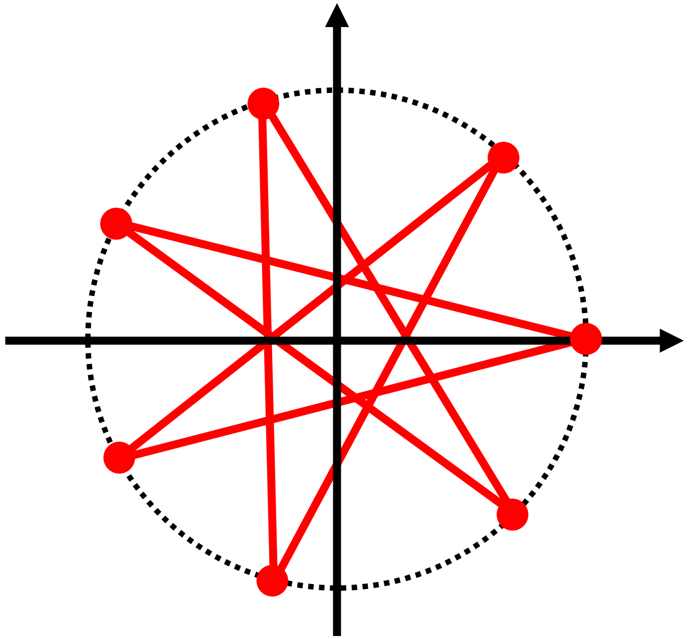
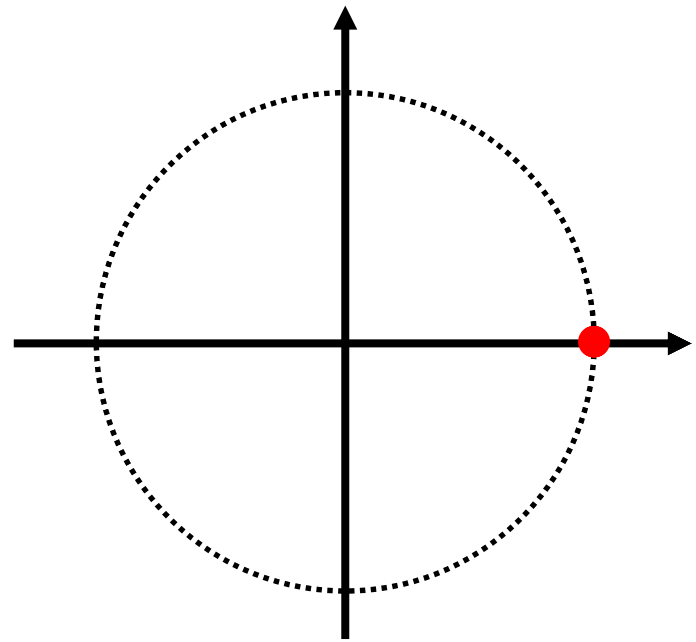


Low and high frequencies

Visualisation of the last row.

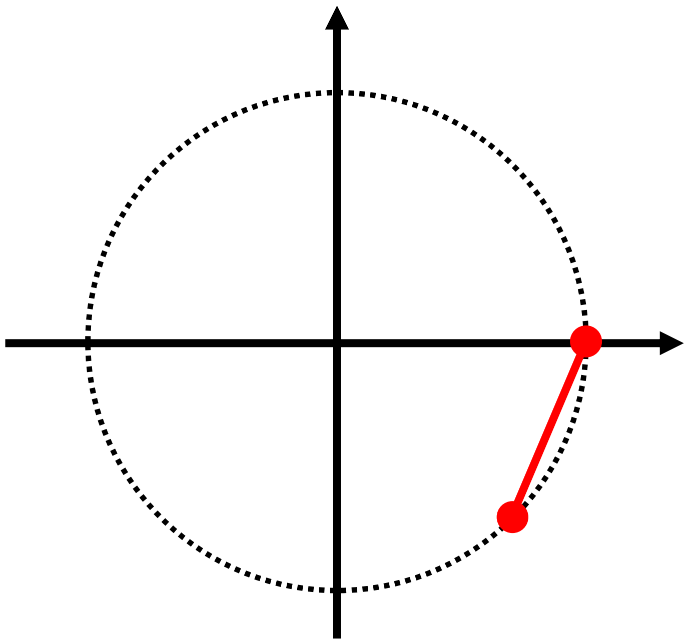
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the last row.

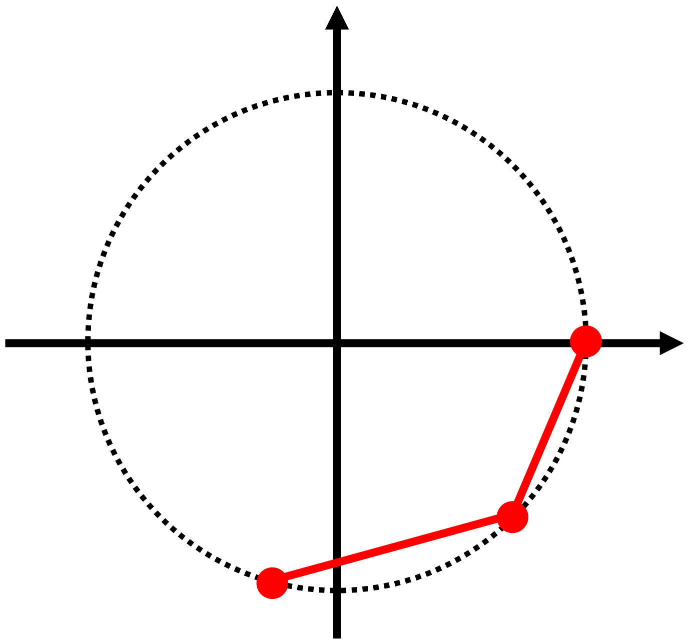
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the last row.

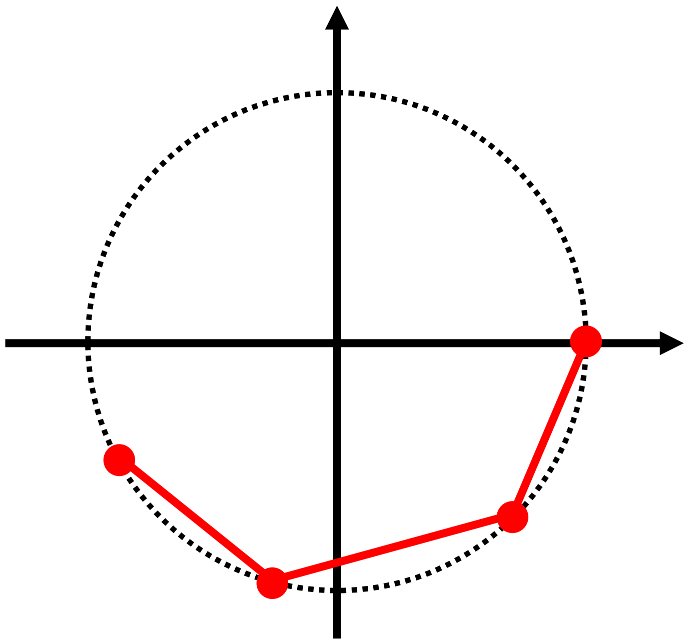
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



Low and high frequencies

Visualisation of the last row.

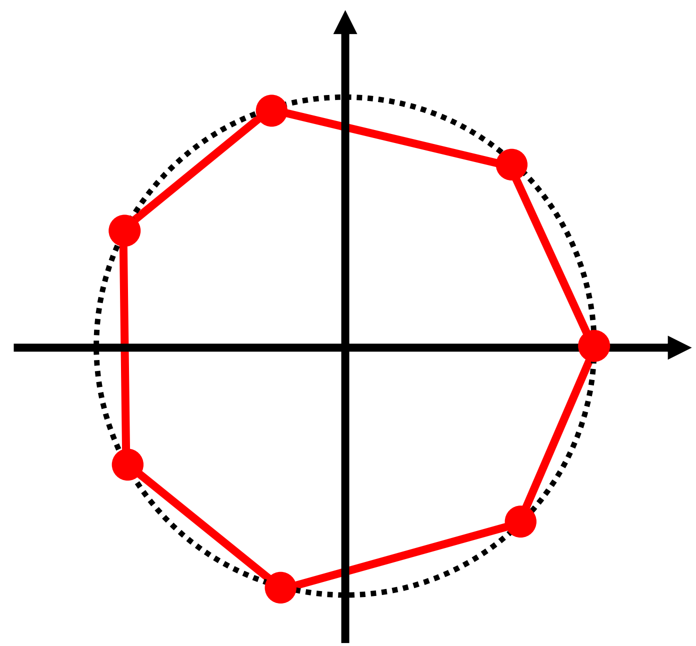
$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\
1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\
1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\
1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\
1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36}
\end{bmatrix}
$$



Low and high frequencies

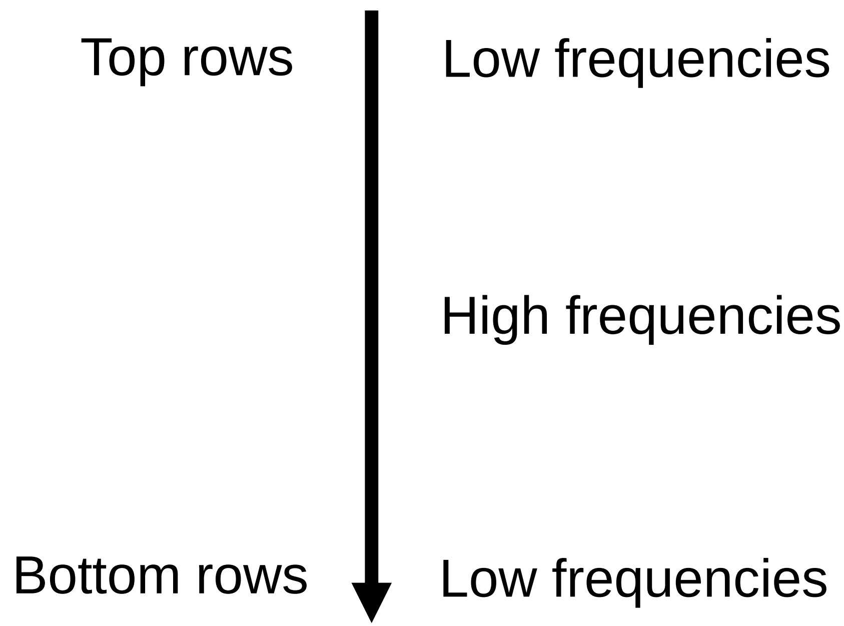
The visualisation of the last row. We made one full clockwise circle.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} \end{bmatrix}$$



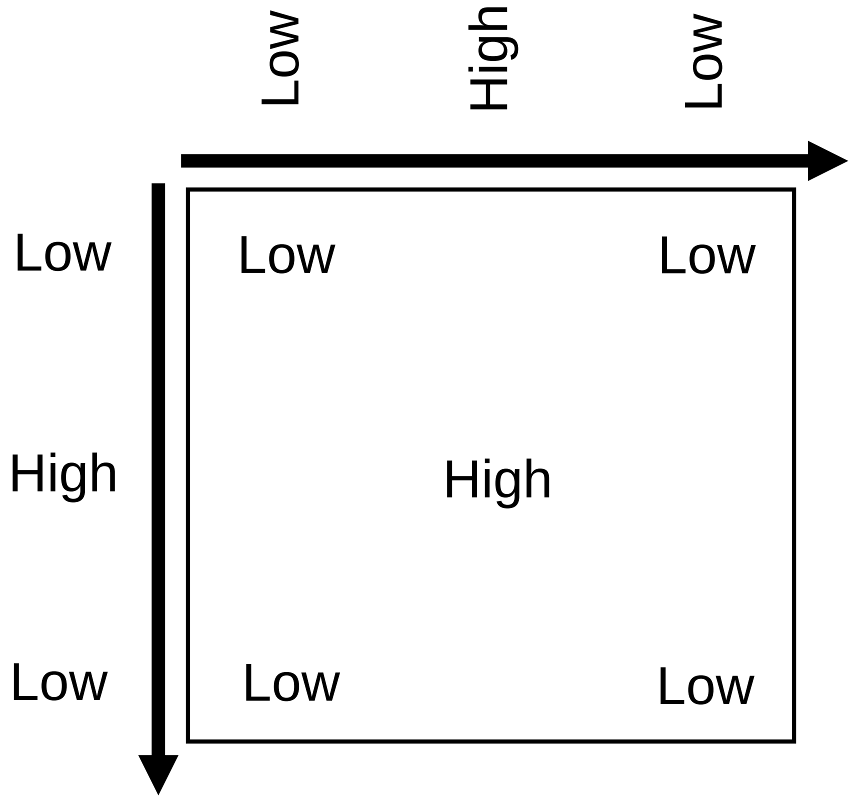
Low and high frequencies

Frequencies of the 1D DFT



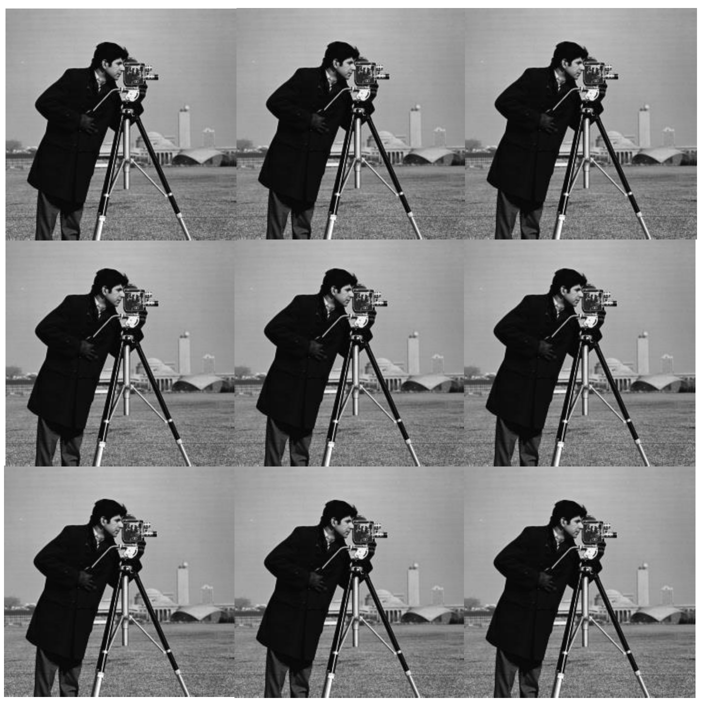
Low and high frequencies

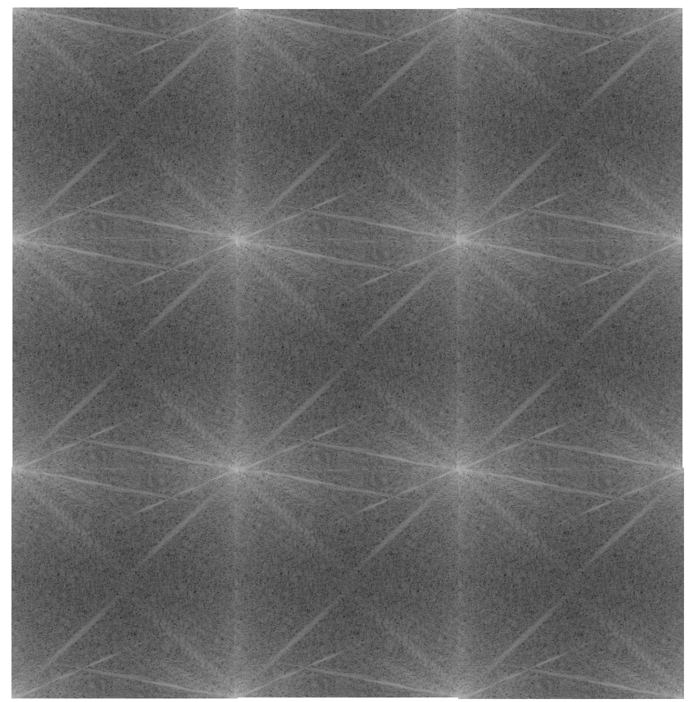
Frequencies of the 2D DFT



Spectrum shift

A doubly periodic image has a doubly periodic discrete Fourier transform.

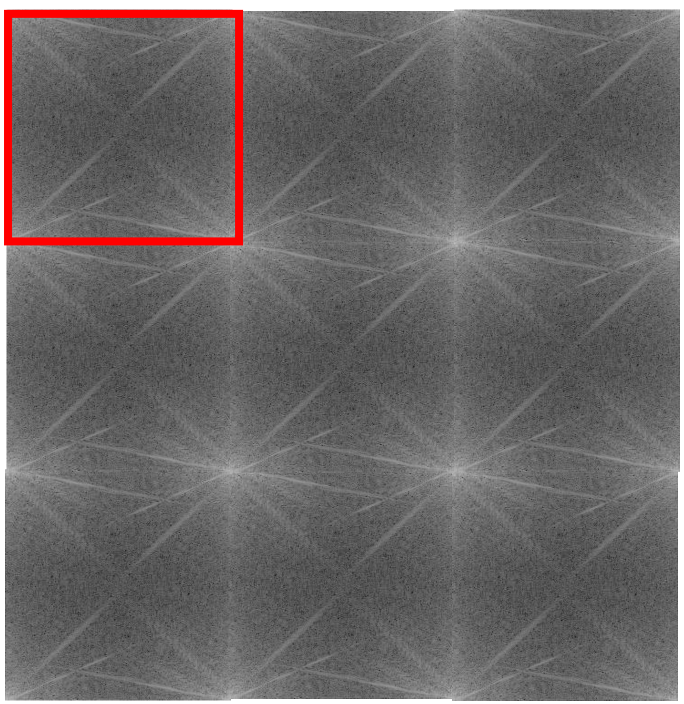


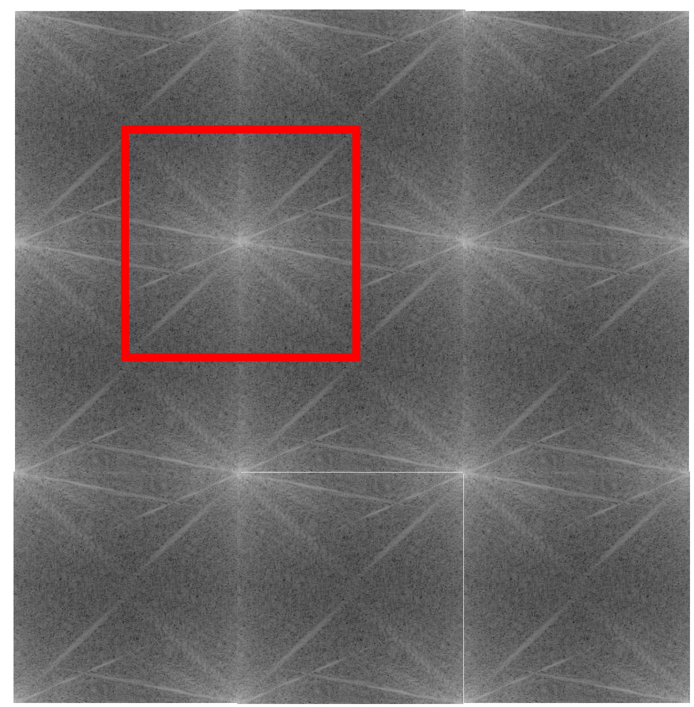


Spectrum shift

We shift the origin of the transform to the centre.

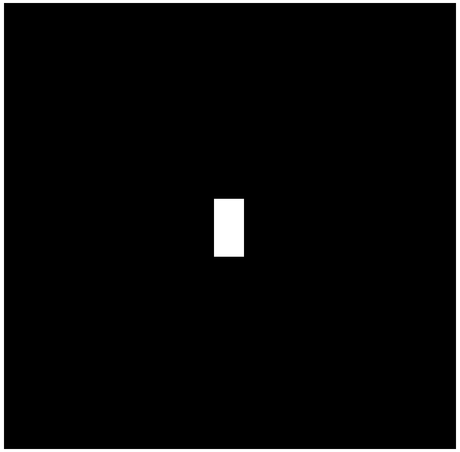
Now the low frequency information is in the centre of the DFT.



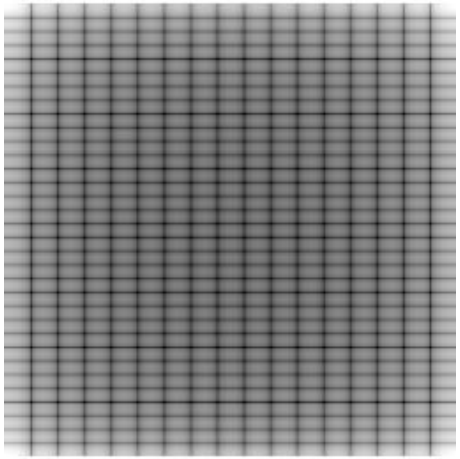


Spectrum shift

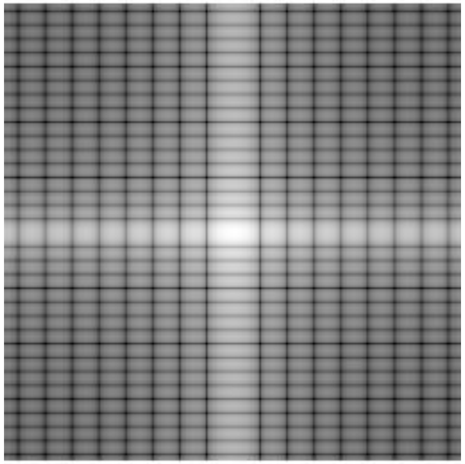
Original image



Log enhanced
transform



Shifted
log enhanced
transform



Overview

Discrete Fourier Transform (DFT)

The spectrum of DFT

The convolution theorem

Filtering in the frequency domain

The convolution theorem

Let f be an image and h be a filter. If necessary, we assume that any of them or both are extended with 0's beyond their boundary, so they have the same dimension.

Let F and H be the corresponding DFT transforms.

The convolution theorem

The convolution theorem states that:

$$f(x,y) \otimes h(x,y) \Leftrightarrow H(u,v) \bullet F(u,v)$$

The symbol $\otimes$ on the left hand side denotes convolution. That is, the usual linear filtering of the image *f* by the filter *h*, as described earlier in the course.

The multiplication $\bullet$ in the right hand side is component-wise (not matrix multiplication).

Overview

Discrete Fourier Transform (DFT)

The spectrum of DFT

The convolution theorem

Filtering in the frequency domain

Frequency domain filters

By finding the DFT of a spatial linear filter we can create an equivalent frequency domain filter.

By the convolution theorem, filtering in the frequency domain requires component-wise multiplication instead of matrix convolution.

Frequency domain filters

If a spatial filter is large, instead of matrix convolution, it is computationally more efficient to:

Find the DFT of the image and the filter

Do the filtering in the frequency domain

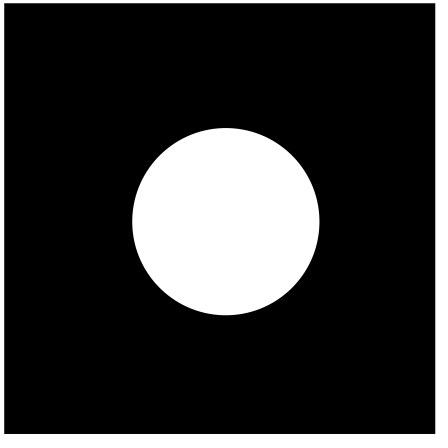
Inverse the DFT to go back to the spatial domain.

Ideal lowpass filter

We can also directly design filters for the frequency domain.

Assume that the frequency domain has been shifted and the low frequencies are at the centre of the transform.

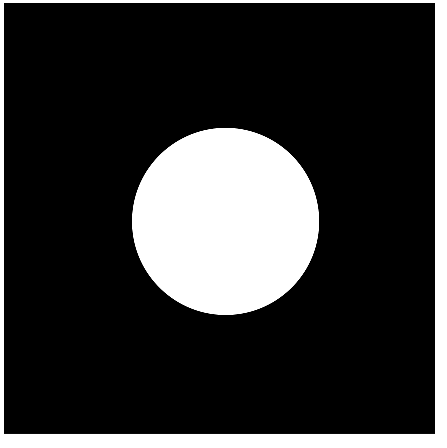
Consider the filter *H* with values 1 near the centre of the image and values 0 further from the centre.



Ideal lowpass filter

This filter retains the low frequencies and eliminates the high frequencies.

Remember that the filter *H(u,v)* acts on the transform *F(u,v)* by component-wise multiplication $H(u,v) \bullet F(u,v)$



Ideal lowpass filters

This filter is called the ideal lowpass filter.

The exact definition of the ideal lowpass filter is

$$H(u,v) = \begin{cases} 1 & \text{if} \quad d(u,v) \leq d_0 \\ 0 & \text{if} \quad d(u,v) > d_0 \end{cases}$$

where *d(u,v)* is the distance of *(u,v)* from the centre of the filter and d_0 is a positive number (the radius of the white circle).

Lowpass filters

The ideal lowpass filter is discontinuous. Frequencies just inside the white circle are preserved, while neighbouring frequencies just outside the white circle are completely eliminated.

The Butterworth lowpass filter of order *n* addresses this problem. It is described by the equation:

$$H(u,v) = \frac{1}{1 + (d(u,v)/d_0)^{2n}}$$

The Gaussian lowpass filter is also smooth:

$$H(u,v) = e^{-d^2(u,v)/d_0}$$

Ideal highpass filters

In some applications we are interested in the details of the image. For example, we might want to enhance a blurred image.

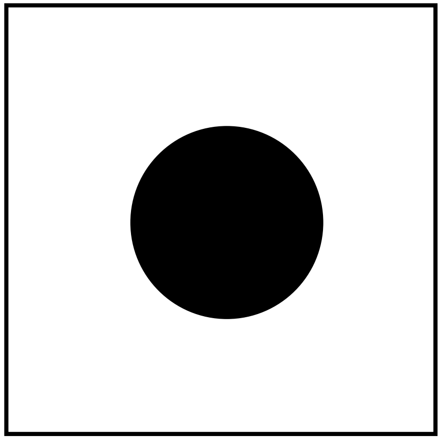
The details of the image correspond to the high frequencies of the DFT and can be extracted by a highpass filter.

We can construct a highpass filter from a lowpass:

$$H_{\mathrm{highpass}} = 1 - H_{\mathrm{lowpass}}$$

Highpass filters

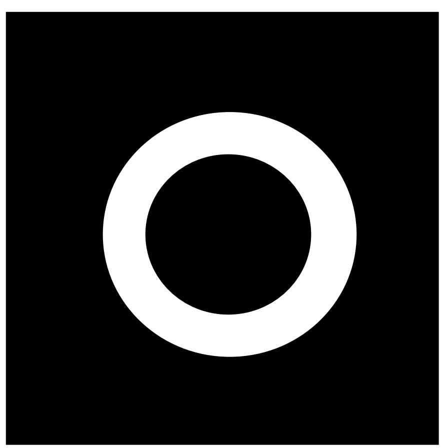
The ideal highpass filter



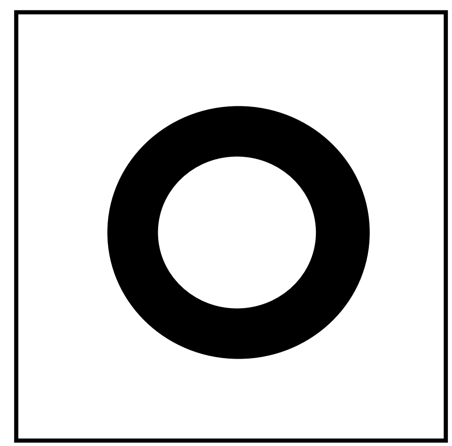
Other ideal filters

The ideal bandpass filter retains the frequencies inside a given band and eliminates all the other.

The ideal bandreject filter eliminates the frequencies inside a given band and retains all the other.



Ideal bandpass filter



Ideal bandreject filter

Example

A lowpass filter for the DFT frequency domain.